Some characterizations of strongly σ-short Boolean Algebras (Forcing Method and Large Cardinal Axioms)

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Some characterizations of strongly σ-short Boolean Algebras

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Abstract. We give some characterizations of strongly σ-short Boolean algebras.

In this report, we give some characterizations of strongly σ-short Boolean algebras. In [2], we introduced σ-short Boolean algebras and strongly σ-short Boolean algebras. We say that a subset D of a Boolean algebra $\mathbf{B}$ is *dense*, in symbol $D \subset_d \mathbf{B}$, if for every positive element $b \in \mathbf{B}$ there exists $d \in D$ such that $0 < d \leq b$, *σ-short* if every strictly descending sequence of length ω in D does not have a nonzero lower bound in $\mathbf{B}$, *$\wedge$-closed* if for every $d_1, d_2 \in D$ with $d_1 \wedge d_2 > \mathbf{0}$, $d_1 \wedge d_2 \in D$. $\mathbf{B}$ is said to be *σ-short* if it has a σ-short dense subset and *strongly σ-short* if it has a σ-short $\wedge$-closed dense subset. We denote by $d(\mathbf{B})$ the density of $\mathbf{B}$. We assume that Boolean algebras are infinite and atomless in this report.

In [2], it was open whether measure algebras are strongly σ-short. Jörg Brendle showed the following theorem(see [1]).

Theorem A (Brendle). *Let $\mathbf{B}_\kappa$ be the algebra for adding κ many random reals.*

1. *$\mathbf{B}_\omega$ is not strongly σ-short.*
2. *Suppose that $d(\mathbf{B}_\kappa) = \kappa$. Then $\mathbf{B}_\kappa$ is strongly σ-short.*

We say that a Boolean algebra $\mathbf{B}$ has *(κ, ω)-caliber* if for any uncountable subset $T \subseteq \mathbf{B}$ of size κ, there is countable $F \subseteq T$ such that F has a non-zero lower bound in $\mathbf{B}$. It is well-known that the random algebra has (ω_1, ω)-caliber.

Y. Yoshinobu and I extended the first result above more general as follows(see [1]).

Theorem B (Takahashi-Yoshinobu). *Suppose that $\mathbf{B}$ satisfies (κ, ω)-caliber and $d(\mathbf{B}) \geq \kappa$. Then $\mathbf{B}$ is not strongly σ-short.*

In this report, we extend these theorems and give some characterizations of strongly σ-short Boolean algebras.

For $X \subset \mathbf{B}$, let $\bigwedge\!\!\!\bigwedge X = \{x_1 \wedge \cdots \wedge x_n > \mathbf{0} \mid x_1, \cdots, x_n \in X, n \in \omega\}$ and $[X]^\omega = \{Y \subseteq X \mid |Y| = \omega\}$, where $|Y|$ is the cardinality of Y.

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Theorem. *The following are equivalent.*

(1) $\mathbf{B}$ *is strongly* σ*-short.*

(2) *There exists* $X \subset \mathbf{B}$ *such that* $\bigwedge\!\!\!\bigwedge X \subset_d B$ *and* $\bigwedge Y = \mathbf{0}$ *for every* $Y \in [X]^\omega$.

(3) *There exists* $X \subset_d \mathbf{B}$ *such that* $\bigwedge Y = \mathbf{0}$ *for every* $Y \in [X]^\omega$.

(4) *There exist* $X \subset \mathbf{B}, D \subset_d \mathbf{B}$ *and* $f : D \xrightarrow{1-1} X$ *such that* $\bigwedge Y = \mathbf{0}$ *for every* $Y \in [X]^\omega$ *and* $d \wedge f(d) > \mathbf{0}$ *for every* $d \in D$.

(5) *There exists* $X \subset_d \mathbf{B}$ *such that* $\{y \in X | y \geq x\}$ *is finite for every* $x \in X$.

(6) *There exists a sequence* $\{X_n\}_{n\in\omega}$ *of subsets of* $\mathbf{B}$ *which satisfies the following conditions:*

   (a) X_n *is a pairwise incomparable subset of* $\mathbf{B}$.

   (b) *If* $x \in X_n, y \in X_m$ *and* $n < m$, *then* $y \not\geq x$.

   (c) $\{y \in X_m | y \geq x\}$ *is finite for every* $m < n$ *and* $x \in X_n$.

   (d) $X := \bigcup_{n\in\omega} X_n \subset_d \mathbf{B}$

Proof of theorem. (1)⇒(2): Suppose that $\mathbf{B}$ is strongly σ-short. Let D be a σ-short, $\wedge$-closed and dense subset of $\mathbf{B}$. Without loss of generality, we assume that $|D| = d(\mathbf{B})$. Put $\kappa = d(\mathbf{B})$. Let $\{d_\alpha | \alpha < \kappa\}$ be an enumeration of elements of D. We shall find $D^\alpha \subset \mathbf{B}$ and $\Lambda^\alpha \subset \alpha$ for $\alpha < \kappa$ such that

(i) $\forall \alpha < \kappa[\Lambda^\alpha \neq \Lambda^{\alpha+1} \Longrightarrow \Lambda^{\alpha+1} = \Lambda^\alpha \cup \{\alpha\}]$,

(ii) $\forall \alpha < \kappa \exists x \in D^{\alpha+1}[x \leq d_\alpha]$,

(iii) $D^\alpha = \bigwedge\!\!\!\bigwedge \{d_\beta | \beta \in \Lambda^\alpha\}$, and

(iv) $\forall \alpha < \kappa[\alpha \in \Lambda^{\alpha+1} \Leftrightarrow \forall d \in D^\alpha [d \not\leq d_\alpha]]$.

Assuming such D^α and Λ^α may be found, let

$$\Lambda := \bigcup_{\alpha<\kappa} \Lambda^\alpha \quad \text{and} \quad D' := \bigcup_{\alpha<\kappa} D^\alpha.$$

By (ii), D' is a dense subset of $\mathbf{B}$. Put $X = \{d_\alpha | \alpha \in \Lambda\}$. By (iii), $D' = \bigwedge\!\!\!\bigwedge X$, so $\bigwedge\!\!\!\bigwedge X$ is a dense subset of $\mathbf{B}$. Let Y be a countable subset of X and $\{d_{\alpha_n}\}_{n\in\omega}$ be its enumeration such that $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$. We show that $\bigwedge Y = \mathbf{0}$. Without loss of generality, we may assume that for any finite subset Y_0 of Y, $\bigwedge Y_0 > \mathbf{0}$. Put $e_n := d_{\alpha_0} \wedge \cdots \wedge d_{\alpha_n}$ for every $n \in \omega$. Since $\alpha_n \in \Lambda$, by (i), (iii) and (iv), we have $\alpha_n \in \Lambda^{\alpha_n+1}$, so that $d_{\alpha_n} \in D^{\alpha_n+1}$ and for every $d \in D^{\alpha_n}$, $d \not\leq d_{\alpha_n}$. Since

$e_{n-1} \in D^{\alpha_{n-1}+1} \subset D^{\alpha_n}$, $e_{n-1} \not\leq d_{\alpha_n}$. So we have $e_0 > e_1 > e_2 > \cdots$. Hence $\{e_n\}_{n\in\omega}$ is a strict decreasing sequence in D. Therefore $\bigwedge Y = \bigwedge_{n\in\omega} e_n = \mathbf{0}$.

We define D^α and Λ^α by induction. Suppose that D^β, Λ^β $(\beta < \alpha)$ are defined. If α is limit, then

$$D^\alpha := \bigcup_{\beta<\alpha} D^\beta \quad \text{and} \quad \Lambda^\alpha := \bigcup_{\beta<\alpha} \Lambda^\beta.$$

If α is successor (say $\alpha_0 + 1$), then we define D^α, Λ^α as follows.

If $\exists d \in D^{\alpha_0}[d \leq d_{\alpha_0}]$, then put

$$D^\alpha := D^{\alpha_0} \quad \text{and} \quad \Lambda^\alpha := \Lambda^{\alpha_0}.$$

If $\forall d \in D^{\alpha_0}[d \not\leq d_{\alpha_0}]$, then put

$$D^\alpha := \bigwedge\!\!\!\bigwedge(D^{\alpha_0} \cup \{d_{\alpha_0}\}) \quad \text{and} \quad \Lambda^\alpha := \Lambda^{\alpha_0} \cup \{\alpha_0\}.$$

It is easy to show that (i), (ii) and (iv) hold. We show (iii) by induction. Suppose that (iii) holds for every $\beta < \alpha$. If α is limit, then

$$D^\alpha = \bigcup_{\beta<\alpha} D^\beta = \bigcup_{\beta<\alpha} \bigwedge\!\!\!\bigwedge\{d_\gamma | \gamma \in \Lambda^\beta\} = \bigwedge\!\!\!\bigwedge\{d_\beta | \beta \in \Lambda^\alpha\}.$$

Suppose that $\alpha = \alpha_0 + 1$. If $\exists d \in D^{\alpha_0}[d \leq d_{\alpha_0}]$, then it is clear that (iii) holds for α. If $\forall d \in D^{\alpha_0}[d \not\leq d_{\alpha_0}]$, then

$$D^\alpha = \bigwedge\!\!\!\bigwedge(D^{\alpha_0} \cup \{d_{\alpha_0}\}) = \bigwedge\!\!\!\bigwedge(\bigwedge\!\!\!\bigwedge\{d_\beta | \beta \in \Lambda^{\alpha_0}\} \cup \{d_{\alpha_0}\}) = \bigwedge\!\!\!\bigwedge\{d_\beta | \beta \in \Lambda^\alpha\}.$$

(2)$\Rightarrow$(3): Easy.

(3)$\Rightarrow$(4): Put $D := X$ and $f := Id_D$.

(4)$\Rightarrow$(1): Put $D_0 := \{d \wedge f(d) | d \in D\}$ and $D_1 := \bigwedge\!\!\!\bigwedge D_0$. Since D_0 is dense in $\mathbf{B}$, D_1 is also dense in $\mathbf{B}$ and $\wedge$-close. To see that D_1 is σ-short, it is enough to show that $\bigwedge Y = \mathbf{0}$ for every $Y \in [D_1]^\omega$. Let $Y := \{d_n \wedge f(d_n) | n \in \omega\}$. Then we have $\bigwedge Y = \bigwedge_{n\in\omega} d_n \wedge \bigwedge_{n\in\omega} f(d_n)$. Since f is one-to-one, $f(d_n) \neq f(d_m)$ for $n \neq m$. Hence $\{f(d_n)\ n \in \omega\} \in [X]^\omega$. Therefore $\bigwedge Y \leq \bigwedge_{n\in\omega} f(d_n) = \mathbf{0}$.

(5) $\Leftrightarrow$ (3): Easy.

(5)$\Rightarrow$(6): Let X be a dense subset of $\mathbf{B}$ such that $\{y \in X | y \geq x\}$ is finite for every $x \in X$. Put $X_n := \{d \in X || \{x \in X | x \geq d\}| = n\}$ for every $n \in \omega$. Then it is easy to show that $\{X_n\}_{n\in\omega}$ satisfies conditions (a)–(d).

(6)$\Rightarrow$(5): Put $X := \bigcup_{n\in\omega} X_n$. Then X is dense in $\mathbf{B}$ by (d). For every $x \in X$, there exists $n \in \omega$ such that $x \in X_n$. Then $\{y \in X | y \geq x\} = \bigcup_{m<n} \{y \in X_m | y \geq x\}$ by (a),(b) and (c). Hence $\{y \in X | y \geq x\}$ is finite. $\square$

Theorem B (Takahashi-Yoshinobu). *Suppose that* $\mathbf{B}$ *satisfies* (κ, ω)*-caliber and* $d(B) \geq \kappa$. *Then* $\mathbf{B}$ *is not strongly* σ*-short.*

Proof of Theorem B: Suppose that $\mathbf{B}$ is strongly σ-short. Then by virtue of main theorem, there exists $X \subset_d B$ such that $\bigwedge Y = \mathbf{0}$ for every $Y \in [X]^\omega$. Since $|T| \geq d(\mathbf{B}) \geq \kappa$, there is countable $F \subseteq T$ such that F has a non-zero lower bound in $\mathbf{B}$. This contradicts that $\mathbf{B}$ satisfies (κ, ω)-caliber. $\square$

Theorem A (Brendle). *Let* $\mathbf{B}_\kappa$ *be the algebra for adding* κ *many random reals.*

1. $\mathbf{B}_\omega$ *is not strongly* σ*-short.*
2. *Suppose that* $d(\mathbf{B}_\kappa) = \kappa$. *Then* $\mathbf{B}_\kappa$ *is strongly* σ*-short.*

Proof: (1): Since $\mathbf{B}_\omega$ satisfies (ω_1, ω)-caliber, $\mathbf{B}_\omega$ is not strongly σ-short by virtue of Theorem B.
(2): Let $D \subseteq B_\kappa$ be dense, $|D| = \kappa$. Say $D = \{b_\alpha;\ \alpha < \kappa\}$. For each α choose $\gamma_\alpha \notin \mathsf{supp}(b_\alpha)$ in such a way that the γ_α are distinct for distinct α. Let $f(b_\alpha) := [\{\langle\langle\gamma_\alpha, 0\rangle, 0\rangle\}]$. Here $\{\langle\langle\gamma_\alpha, 0\rangle, 0\rangle\}$ denotes the partial function $p : \kappa \times \omega \to 2$ with domain the singleton $\{\langle\gamma_\alpha, 0\rangle\}$ and $p(\langle\gamma_\alpha, 0\rangle) = 0$. $[p]$ is the open set defined by p. Then f satisfies the assumption of (4) of the main theorem. Hence $\mathbf{B}_\kappa$ is strongly σ-short. $\square$

Open Problems

1. Are perfect tree forcings, Hechler forcing σ-short?
2. For every σ-short $\mathbf{B}$, does there exist a sequence $\{X_n\}_{n\in\omega}$ of subsets of $\mathbf{B}$ which satisfies the following conditions:
   (a) X_n is a pairwise incomparable subset of $\mathbf{B}$.
   (b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.
   (c) $\bigcup_{n\in\omega} X_n \subset_d B$

References

[1] M. Takahashi, *On Strongly σ-Short Boolean Algebras*, Proceedings of General Topology Symposium held in Kobe, 2002, 74–79

[2] M. Takahashi and Y. Yoshinobu, *σ-short Boolean algebras*, Mathematical Logic Quarterly, Vol. 49 No. 6 (2003), 543–549